

A Supersymmetry approach to billiards with randomly distributed scatterers

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Abstract. The density of states for a chaotic billiard with randomly distributed point-like scatterers is calculated, doubly averaged over the positions of the impurities and the shape of the billiard. Truncating the billiard Hamiltonian to a $N \times N$ matrix, an explicit analytic expression is obtained for the case of broken time-reversal symmetry, depending on rank N of the matrix, number L of scatterers, and strength of the scattering potential. In the strong coupling limit a discontinuous change is observed in the density of states as soon as L exceeds N .

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1. Motivation

Experiments with classical waves have become a very versatile tool to study localization due to disorder. In particular the experiments by Lagendijk and coworkers (Wiersma *et al* 1997) on the localization of light in powders, and by the Genack group on the localization of microwaves in disordered metallic spheres (Chabanov and Genack 2001) have to be mentioned (for a review of these types of experiments see Soukoulis 1996). Moreover microwave techniques are able to study spatially resolved field distributions in disordered systems of linear dimensions in the order of some 10 cm (Kudrolli *et al* 1995, Stöckmann *et al* 2001). Such quantities are inaccessible in electronic quantum dot systems of submicron size (except for the recent experiments by Topinka *et al* (2001)). With increasing frequency one typically observes a transition from localized to delocalized wave functions, depending on the number of scatterers and the strength of the scattering potential. Pulse propagation can be studied as well by microwave techniques as has been shown by Stein *et al* (1995). All quantities of interest are thus experimentally accessible in disordered systems, including conductivity, localization-delocalization transitions, pulse propagation, transition from the ballistic to the diffusive regime, and so on.

On the theoretical side the situation is less favourable. Though there is a vast amount of literature on disordered systems already in the seventies and the eighties of the last century (see e.g. Anderson 1978, Lee and Ramakrishnan 1985 for reviews), there is as yet no theory available covering the complete range from the localized to the delocalized regime. Today the standard approach to study disordered systems uses supersymmetry techniques to arrive at Efetov's non-linear σ model (Efetov 1983). It has the serious draw-back that the occurring supersymmetric variables are field variables depending on the position. Only in the zero mode approximation, where the position dependencies are neglected, the model can be solved exactly and reproduces random matrix theory. This is why localization-delocalization transitions cannot be obtained in this way. Only perturbational corrections are possible, with the consequence that e.g. the distribution of wave function intensities deviates slightly from the Porter-Thomas behaviour found in the delocalized regime (see Guhr *et al* 1998, Mirlin 2000 for reviews).

In this paper an alternative approach is proposed which avoids the complication of position-dependent supersymmetry fields. Moreover, it is even closer to the situation met in experiments, as Efetov's ansatz.

2. The model

Let us consider a billiard system with hard walls and statistically distributed scatterers described by the Hamiltonian

$$H = H_0 + V, \tag{2.1}$$

where H_0 is the operator of kinetic energy with matrix elements

$$(H_0)_{nm} = E_n^0 \delta_{nm} \quad (2.2)$$

and V is the potential energy of the scatterers. In Efetov's approach the potential is assumed to be delta correlated,

$$\langle V(r)V(r') \rangle \sim \delta(r - r'), \quad (2.3)$$

which gives rise to the above mentioned problems with position-dependent supersymmetry fields. In this paper the more explicit ansatz

$$V(r) = 4\pi\lambda \sum_{l=0}^L \delta(r - r_l) \quad (2.4)$$

is used instead, where the r_l are the positions of the scatterers, and L is its number. The factor 4π has been introduced for later convenience. This ansatz dates back to Lifshitz (1964) and has been applied since then by various authors, among others Luttinger *et al* (1983, 1987).

In the basis of eigenfunctions $\psi_n(r)$ of the billiard without scatterers the matrix elements of $V(r)$ read

$$V_{nm} = 4\pi\lambda \sum_{l=0}^L \psi_n^*(r_l) \psi_m(r_l). \quad (2.5)$$

To simplify the calculations we assume that time-reversal symmetry is broken, e. g. due to the presence of an applied magnetic field.

We are now going to calculate the density of states, averaged over the positions of the impurities,

$$\langle \rho(E) \rangle = -\frac{1}{\pi} \text{Im} \left\langle \text{Tr} \left(\frac{1}{E_+ - H} \right) \right\rangle, \quad (2.6)$$

where $E_+ = E + i\epsilon$. Using standard transformations, equation (2.6) can be written as

$$\rho(E) = -\frac{1}{\pi} \frac{d}{dE'} \text{Im} [Z(E, E')] \Big|_{E'=E}, \quad (2.7)$$

where

$$Z(E, E') = \left\langle \frac{|E'_+ - H|}{|E_+ - H|} \right\rangle. \quad (2.8)$$

Z may be written in terms of an integral over commuting and anti-commuting variables as (Verbaarschot *et al* 1985)

$$Z = \left\langle \int d[x] \exp \left(i \sum_{\alpha\beta} \left[(E_+ \delta_{\alpha\beta} - H_{\alpha\beta}) x_\alpha^* x_\beta + (E'_+ \delta_{\alpha\beta} - H_{\alpha\beta}) \xi_\alpha^* \xi_\beta \right] \right) \right\rangle, \quad (2.9)$$

where

$$d[x] = \prod_{\alpha=1}^N dx_{\alpha}^* dx_{\alpha} d\xi_{\alpha}^* d\xi_{\alpha} \quad (2.10)$$

We adopt the usual convention and use Latin letters for commuting and Greek ones for anti-commuting variables. In short-hand matrix notation equation (2.9) may be written as

$$Z = \left\langle \int d[x] e^{i\mathbf{x}^\dagger (\mathbf{E} - \mathbf{H}) \mathbf{x}} \right\rangle, \quad (2.11)$$

where

$$\mathbf{x} = (x_1, \xi_1, \dots, x_N, \xi_N)^T, \quad (2.12)$$

$$\mathbf{E} = E \otimes \mathbf{1}_N = \begin{pmatrix} E & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E \end{pmatrix}, \quad E = \begin{pmatrix} E_+ & \cdot \\ \cdot & E'_+ \end{pmatrix}, \quad (2.13)$$

and

$$\mathbf{H} = \mathbf{1} \otimes H = \begin{pmatrix} H_{11} \mathbf{1} & \cdots & H_{1N} \mathbf{1} \\ \vdots & \ddots & \vdots \\ H_{N1} \mathbf{1} & \cdots & H_{NN} \mathbf{1} \end{pmatrix}. \quad (2.14)$$

In Equations (2.13) and (2.14) $\mathbf{1}_N$ and $\mathbf{1}$ denote the N - and the two-dimensional unit matrix, respectively. Inserting expression (2.1) for H , equation (2.11) reads

$$\begin{aligned} Z &= \int d[x] e^{i\mathbf{x}^\dagger (\mathbf{E} - \mathbf{H}_0) \mathbf{x}} \left\langle e^{-4\pi i \lambda \sum_{l\alpha\beta} \psi_{\alpha}^*(r_l) \psi_{\beta}(r_l) (x_{\alpha}^* x_{\beta} + \xi_{\alpha}^* \xi_{\beta})} \right\rangle \\ &= \int d[x] e^{i\mathbf{x}^\dagger (\mathbf{E} - \mathbf{H}_0) \mathbf{x}} M^L, \end{aligned} \quad (2.15)$$

where

$$M = \left\langle e^{-4\pi i \lambda \sum_{\alpha\beta} \psi_{\alpha}^*(r) \psi_{\beta}(r) (x_{\alpha}^* x_{\beta} + \xi_{\alpha}^* \xi_{\beta})} \right\rangle. \quad (2.16)$$

The average in equation (2.16) has to be taken over the positions of the impurities. But, *and this is the central idea of this paper*, instead of varying over the positions, we may equally well calculate this average by weighting the expression on the right hand side of the equation with the joint probability density $p(\psi_{1R}, \psi_{1I}, \dots)$ to find at any point in the billiard the values $\psi_{1R}, \psi_{1I}, \dots$ for the wave function amplitudes. If the billiard without scatterers is chaotic, the probability density factorizes, $p(\psi_{1R}, \psi_{1I}, \dots) = \prod_{\alpha} p(\psi_{\alpha R}) p(\psi_{\alpha I})$, and real and imaginary part of the wave functions are Gaussian distributed,

$$p(\psi_R) = \sqrt{\frac{A}{\pi}} e^{-A\psi_R^2}, \quad p(\psi_I) = \sqrt{\frac{A}{\pi}} e^{-A\psi_I^2}, \quad (2.17)$$

where A is the billiard area. The average (2.16) over the impurity positions may hence be written as

$$M = \int \prod_{\alpha} [d\psi_{\alpha R} d\psi_{\alpha I} p(\psi_{\alpha R}) p(\psi_{\alpha I})] e^{-4\pi i \lambda \sum_{\alpha\beta} \psi_{\alpha}^* \psi_{\beta} (x_{\alpha}^* x_{\beta} + \xi_{\alpha}^* \xi_{\beta})}. \quad (2.18)$$

With the weight functions (2.17) the integrations are easily performed yielding

$$M = \frac{1}{|\mathbf{1}_N + i \frac{4\pi\lambda}{A} X|}, \quad (2.19)$$

where X is the $N \times N$ matrix with the elements

$$X_{\alpha\beta} = x_{\alpha}^* x_{\beta} + \xi_{\alpha}^* \xi_{\beta}. \quad (2.20)$$

According to the Weyl formula the mean density of states in two-dimensional billiards is given by $\langle \rho \rangle = A/4\pi$. Following the usual practice we normalize this quantity to one, and omit the factor $4\pi/A$ in the following. The determinant (2.19) is now transformed by means of the relation

$$|\mathbf{1}_N + AB| = |\mathbf{1}_M + BA| \quad (2.21)$$

holding for arbitrary $N \times M$ matrices A , and $M \times N$ matrices B . This follows in a straightforward manner from the relation $|M| = \exp[\text{Tr}(\ln M)]$. It is not necessary that the matrices are quadratic providing us with an efficient tool to reduce the rank of determinants. Applied to equation (2.19) relation (2.21) yields

$$M = \frac{1}{|\mathbf{1} + i\lambda \hat{X}|} \quad (2.22)$$

where \hat{X} is the 2×2 supermatrix

$$\hat{X} = \begin{pmatrix} \sum_{\alpha} x_{\alpha} x_{\alpha}^* & \sum_{\alpha} x_{\alpha} \xi_{\alpha}^* \\ \sum_{\alpha} \xi_{\alpha} x_{\alpha}^* & \sum_{\alpha} \xi_{\alpha} \xi_{\alpha}^* \end{pmatrix}. \quad (2.23)$$

We have thus arrived at the intermediate result

$$Z = \int d[x] e^{i\mathbf{x}^{\dagger}(\mathbf{E} - \mathbf{H}_0)\mathbf{x}} |\mathbf{1} + i\lambda \hat{X}|^{-L} \quad (2.24)$$

Whenever there are super matrices involved, determinants and traces have to be interpreted as super determinants and super traces, respectively, where we shall use the convention of Verbaarschot *et al* 1985.

It is instructive to consider the small λ limit of expression (2.24). The determinant may be expanded as

$$\begin{aligned} |\mathbf{1} + i\lambda \hat{X}|^{-L} &= \exp \left[-L \text{Tr} \ln (\mathbf{1} + i\lambda \hat{X}) \right] \\ &= \exp \left[-iL\lambda \text{Tr} \hat{X} - \frac{L\lambda^2}{2} \text{Tr} \hat{X}^2 + \dots \right]. \end{aligned} \quad (2.25)$$

Stopping at the quadratic term, equation (2.24) reads

$$Z = \int d[x] e^{\left[i\mathbf{x}^\dagger (\mathbf{E} - \mathbf{H}_0 - L\lambda \mathbf{1}_N) \mathbf{x} - \frac{L\lambda^2}{2} \text{Tr } \hat{X}^2 \right]}, \quad (2.26)$$

where $\text{Tr } \hat{X} = \mathbf{x}^\dagger \mathbf{x}$ was used. This is exactly the expression obtained for the ensemble average of the Hamiltonian

$$H = H_0 + L\lambda \mathbf{1}_N + H_1. \quad (2.27)$$

where the matrix elements of H_1 are Gaussian random variables with variance $\langle H_1^2 \rangle = L\lambda^2$. We thus can note already at this early stage that in the small λ limit random matrix results will be recovered.

3. The x integrations

The usual approach to perform integrations of type (2.26) is a Hubbard-Stratonovich transformation to remove the $\text{Tr } \hat{X}^2$ term in the exponent, depending on the integration variables in the fourth order. As a result the x integrations reduce to simple Fresnel integrals which are trivially solved.

For the integral (2.24) a Hubbard-Stratonovich transformation is not possible. One way to proceed further is to write again the determinant in terms of a superintegral,

$$\frac{1}{|\mathbf{1} + i\lambda \hat{X}|} = \int d[y] e^{-\mathbf{y}^\dagger (\mathbf{1} + i\lambda \hat{X}) \mathbf{y}}. \quad (3.1)$$

We need L replicas of this integral since the determinant enters in the L th power, leading to the introduction of $4L$ new integration variables. The x integrations can then be performed in the usual manner.

To avoid the introduction of such a large number of new integration variables, we apply another approach. Let us consider the integral

$$I_L(A) = \int d[t] |T|^L e^{-\text{Tr}(AT)}, \quad (3.2)$$

where

$$A = \begin{pmatrix} a & \alpha^* \\ \alpha & \bar{a} \end{pmatrix}, \quad T = \begin{pmatrix} t & \tau^* \\ \tau & \bar{t} \end{pmatrix} \quad (3.3)$$

are supermatrices of rank 2.

Because of the basis independence of trace and determinant it is always possible to chose the T variables such that A is diagonal, i.e. $\alpha = \alpha^* = 0$. Equation (3.2) then reads

$$I_L(A) = \int dt d\bar{t} d\tau^* d\tau \left| \begin{array}{cc} t & \tau^* \\ \tau & \bar{t} \end{array} \right|^L e^{-(at - \bar{a}\bar{t})}, \quad (3.4)$$

Introducing new integration variables $s = at$, $\bar{s} = \bar{a}\bar{t}$, $\sigma^* = a\tau^*$, $\sigma = \bar{a}\tau$ we obtain

$$\begin{aligned}
 I_L(A) &= \int \frac{ds}{|a|} \frac{d\bar{s}}{|\bar{a}|} |a| d\sigma^* |\bar{a}| d\sigma \left| \begin{array}{cc} a^{-1}s & a^{-1}\sigma^* \\ \bar{a}^{-1}\sigma & \bar{a}^{-1}\bar{s} \end{array} \right|^L e^{-\text{Tr}(s-\bar{s})} \\
 &= \left(\frac{|\bar{a}|}{|a|} \right)^L I_L,
 \end{aligned} \tag{3.5}$$

where

$$I_L = \int d[t] |T|^L e^{-\text{Tr}(T)}, \tag{3.6}$$

or

$$|A|^{-L} = I_L(A)/I_L. \tag{3.7}$$

Because of the basis independence of this expression the latter result holds for arbitrary supermatrices A , not necessarily diagonal.

This is an alternative to equation (3.1) to express the power of a determinant as a superintegral, avoiding the need to introduce L replicas. The question arises, how the paths of integration are to be chosen to make the integral (3.2) well-defined. From equation (3.3) we have

$$\text{Tr } T = t - \bar{t}, \quad |T| = \left(t - \frac{\tau^* \tau}{\bar{t}} \right) / \bar{t}. \tag{3.8}$$

Shifting the variable t by $\tau^* \tau / \bar{t}$, integral (3.6) reads

$$I_L = \int dt d\bar{t} d\tau^* d\tau \left| \frac{t}{\bar{t}} \right|^L e^{-(t+\tau^* \tau / \bar{t} - \bar{t})}. \tag{3.9}$$

The integration over the antisymmetric variables is straightforward, and we are left with the t, \bar{t} integrations,

$$I_L = \frac{1}{2\pi} \int dt d\bar{t} \frac{t^L}{\bar{t}^{L+1}} e^{-(t-\bar{t})}. \tag{3.10}$$

Let us assume for a moment that L is non-integer. Then we may define an integration path starting at $e^{i\phi}\infty$, encircling the origin counterclockwise, and returning to $e^{i\phi}\infty$. The phase angle ϕ has to be chosen in a way that the integral is well-defined. We end thus with the well-known integral representation for the reciprocal Gamma function, both for the t and the \bar{t} integration, with the result

$$\begin{aligned}
 I_L &= \frac{2\pi}{\Gamma(L+1)\Gamma(-L)} e^{i\pi L} = 2 \sin \pi(L+1) e^{i\pi L} \\
 &= i \left(e^{2\pi i L} - 1 \right).
 \end{aligned} \tag{3.11}$$

Equation (3.7) thus is well-defined for non-integer L if the paths of integration are chosen as described above. For integer L the expression on the right hand side is not

defined, but it is easily seen that the limit (non-integer L) \rightarrow (integer L) exists and gives

$$|A|^{-L} = \frac{1}{i} \int_0^{e^{i\phi}\infty} dt \oint d\bar{t} d\tau^* d\tau |T|^L e^{-\text{Tr}(AT)}, \quad (3.12)$$

where the \bar{t} integration is performed counterclockwise on a circle about the origin. Equation (3.12) holds for all natural numbers L .

Applied to equation (2.24), we have

$$Z = \frac{1}{I_L} \int d[x] d[t] e^{i\mathbf{x}^\dagger(\mathbf{E}-\mathbf{H}_0)\mathbf{x}} |T|^L e^{-\text{Tr}[T(\mathbf{1}+i\lambda\hat{X})]} \quad (3.13)$$

Now the x integrations can be performed, using definition (2.23),

$$\begin{aligned} \int d[x] e^{i\mathbf{x}^\dagger(\mathbf{E}-\mathbf{H}_0)\mathbf{x}-\text{Tr}(\lambda T\hat{X})} &= \int d[x] e^{i\mathbf{x}^\dagger(\mathbf{E}-\mathbf{H}_0-\lambda T\mathbf{1}_N)\mathbf{x}} \\ &= \prod_\alpha \frac{1}{|E - E_a^0\mathbf{1} - \lambda T|}, \end{aligned} \quad (3.14)$$

whence follows

$$Z = \frac{1}{I_L} \int d[t] |T|^L e^{-\text{Tr} T} \prod_\alpha \frac{1}{|E - E_a^0\mathbf{1} - \lambda T|}. \quad (3.15)$$

This may alternatively be written as

$$Z = \frac{1}{I_L} \int d[t] e^{-\text{Tr}[F(T)]}, \quad (3.16)$$

where

$$F(T) = T - L \ln T + \sum_\alpha \ln(E - E_a^0\mathbf{1} - \lambda T). \quad (3.17)$$

Equations (3.15) to (3.17) constitute our next intermediate result. They allow to calculate the averaged density of states of a billiard with randomly distributed scatterers in terms of the eigenenergies of the billiard without scatterers. All integrals can be solved exactly by means of the residuum method. In the remaining step the limit $N \rightarrow \infty$ has to be performed. The occurring infinite products diverge as a consequence of the delta-like singularities in the potential. But the divergencies can be handled in a standard way by a renormalization of the coupling constant (Albeverio and Šeba 1991). See also Bogomolny *et al* 2001, where the situation of a single scatterer in a rectangular billiard is studied.

From microwave experiments it is known, but for systems with time-reversal symmetry only, that in billiards with randomly distributed scatterers the wave functions are localized at low energies, but become delocalized at sufficiently high energies (Kudrolli *et al* 1995, Stöckmann *et al* 2001). A calculation of the averaged density of states as a function of energy from equation (3.15), and of two-point correlation

function, inverse participation ratio etc. from its generalization should thus exhibit clear fingerprints of the localization-delocalization transition.

This will be the program for future works. For the moment let us proceed along a more convenient route by taking N fixed and finite, and by performing a second average over the shape of the billiard.

4. The average over the billiard shape

According to a conjecture of Bohigas, Giannoni, Schmit (1984) the spectrum of a billiard with broken time-reversal symmetry should obey the same statistical features as the spectrum of a random matrix taken from the Gaussian Unitary Ensemble (GUE). Taken this for granted we may replace H_0 in equation (3.13) by a GUE matrix and perform a Gaussian average over the matrix elements to obtain the average over the billiard shape. (Up to now H_0 had been assumed to be diagonal, but because of the basis invariance of the expression we may take any other basis as well; it is much easier to perform the average over the matrix elements than over the eigenvalues.)

The Gaussian average over the matrix elements is trivial and yields

$$\begin{aligned} \langle e^{-i\mathbf{x}^\dagger \mathbf{H}_0 \mathbf{x}} \rangle &= \left\langle e^{-i \sum_{\alpha\beta} (H_0)_{\alpha\beta} (x_\alpha^* x_\beta + \xi_\alpha^* \xi_\beta)} \right\rangle \\ &= \left\langle e^{-\frac{1}{2} \langle (H_0)^2 \rangle \sum_{\alpha\beta} (x_\alpha^* x_\beta + \xi_\alpha^* \xi_\beta)(x_\beta^* x_\alpha + \xi_\beta^* \xi_\alpha)} \right\rangle \\ &= e^{-\frac{N}{2\pi^2} \text{Tr}(\hat{X})^2}. \end{aligned} \quad (4.1)$$

Following common practise again we have shifted the average energy to zero, and have applied the normalization $\langle (H_0)^2 \rangle = N/\pi^2$ yielding a mean density of states of one at $E = 0$ (Verbaarschot *et al* 1985). After a subsequent Hubbard-Stratonovich transformation equation (4.1) reads

$$\langle e^{-i\mathbf{x}^\dagger \mathbf{H}_0 \mathbf{x}} \rangle = \int d[y] e^{-\frac{\pi^2}{2N} \text{Tr} Y^2 - i \text{Tr}(\hat{X} Y)}, \quad (4.2)$$

where

$$Y = \begin{pmatrix} y & \eta^* \\ \eta & \bar{y} \end{pmatrix}. \quad (4.3)$$

To make expression (4.2) well-defined, the y integration has to be performed from $-\infty$ to ∞ , and the \bar{y} integration from $-i\infty$ to $i\infty$. Inserting expression (4.2) into equation (3.13) we get as the result of the shape averaging

$$\begin{aligned} \langle Z \rangle &= \frac{1}{I_L} \int d[x] d[t] d[y] |T|^L e^{-\text{Tr} T} e^{-\frac{\pi^2}{2N} \text{Tr} Y^2} e^{i\mathbf{x}^\dagger (\mathbf{E} - \lambda T \mathbf{1}_N) \mathbf{x}} e^{-i \text{Tr}(\hat{X} Y)} \\ &= \frac{1}{I_L} \int d[t] d[y] |T|^L e^{-\text{Tr} T} e^{-\frac{\pi^2}{2N} \text{Tr} Y^2} \int d[x] e^{i\mathbf{x}^\dagger (E - \lambda T - Y) \mathbf{1}_N \mathbf{x}}, \end{aligned} \quad (4.4)$$

where we have used $\text{Tr}(\hat{X}Y) = \mathbf{x}^\dagger Y \mathbf{1}_N \mathbf{x}$. The X integrations are straightforward and yield

$$\langle Z \rangle = \frac{1}{I_L} \int d[t] d[y] |T|^L e^{-\text{Tr} T} e^{-\frac{\pi^2}{2N} \text{Tr} Y^2} \frac{1}{|E - \lambda T - Y|^N}. \quad (4.5)$$

Again we apply expression (3.7) to rewrite the determinant,

$$\frac{1}{|E - \lambda T - Y|^N} = \frac{1}{I_N} \int d[s] |S|^N e^{\text{Tr} S(E - \lambda T - Y)}, \quad (4.6)$$

where

$$S = \begin{pmatrix} s & \sigma^* \\ \sigma & \bar{s} \end{pmatrix}. \quad (4.7)$$

In addition, we replace T by NT , and Y by NY and obtain

$$\begin{aligned} \langle Z \rangle &= \frac{1}{I_L I_N} \int d[t] d[y] d[s] |T|^L e^{-N \text{Tr} T} e^{-N \frac{\pi^2}{2} \text{Tr} Y^2} |S|^N e^{\text{Tr} S(E - \lambda NT - NY)} \\ &= \frac{1}{I_N} \int d[s] |S|^N e^{\text{Tr}(SE)} \frac{1}{I_L} \int d[t] |T|^L e^{-N \text{Tr} T(\mathbf{1} + \lambda S)} \int d[y] e^{-N \left(\frac{\pi^2}{2} \text{Tr} Y^2 + \text{Tr}(SY) \right)}. \end{aligned} \quad (4.8)$$

The T and the Y integrations can now be performed with the result

$$\langle Z \rangle = \frac{1}{I_N} \int d[s] e^{\text{Tr}(SE)} \frac{|S|^N}{|\mathbf{1} + \lambda S|^L} e^{\frac{N}{2\pi^2} \text{Tr} S^2}. \quad (4.9)$$

Equation (4.9) is the main result of this paper. It is surprisingly simple and allows an easy calculation of the density of states of the billiard with randomly scatterers, doubly averaged over disorder and shape of the billiard. We only have to perform the remaining integrations over 4 commuting and anti-commuting variables.

5. The density of states

The calculation of the integral is easiest, if we transform the matrix S into a diagonal matrix via

$$S = \begin{pmatrix} s & \sigma^* \\ \sigma & \bar{s} \end{pmatrix} = \begin{pmatrix} \sqrt{1 + \beta\gamma} & -\beta \\ -\gamma & \sqrt{1 + \gamma\beta} \end{pmatrix} \begin{pmatrix} s_B & \cdot \\ \cdot & s_F \end{pmatrix} \begin{pmatrix} \sqrt{1 + \beta\gamma} & \beta \\ \gamma & \sqrt{1 + \gamma\beta} \end{pmatrix}. \quad (5.1)$$

After performing the matrix multiplications we have

$$\begin{aligned} s &= s_B + \beta\gamma(s_B - s_F), & \sigma^* &= \beta(s_B - s_F), \\ \sigma &= -\gamma(s_B - s_F), & \bar{s} &= s_F + \beta\gamma(s_B - s_F), \end{aligned} \quad (5.2)$$

whence follows for the volume element

$$d[s] = -\frac{ds_B ds_F d\beta d\gamma}{(s_B - s_F)^2}. \quad (5.3)$$

Remembering that the s_B integration is from 0 to $e^{i\phi}\infty$ with a suitably chosen phase angle ϕ , and that the s_F integration is along a circle about the origin (see equation (3.12)), we obtain from equation (4.9)

$$\begin{aligned} \langle Z \rangle = & -\frac{1}{i} \int_0^{e^{i\phi}\infty} \oint \frac{ds_B ds_F d\beta d\gamma}{(s_B - s_F)^2} e^{E_+[s_B + \beta\gamma(s_B - s_F)] - E'_+[s_F + \beta\gamma(s_B - s_F)]} \\ & \times \left| \frac{s_B}{s_F} \right|^N \left| \frac{1 + \lambda s_F}{1 + \lambda s_B} \right|^L e^{\frac{N}{2\pi^2}(s_B^2 - s_F^2)}. \end{aligned} \quad (5.4)$$

The value for the phase angle can be inferred from equation (4.6): since t and y are real, and E_+ has an infinitesimally small positive imaginary part, the integration has to be performed from 0 to $i\infty$.

The integral over the antisymmetric variables is easily done and yields

$$\begin{aligned} \langle Z \rangle = & \frac{1}{2\pi i} \int_0^{i\infty} ds_B \oint ds_F \frac{E_+ - E'_+}{s_B - s_F} e^{E_+ s_B - E'_+ s_F} \\ & \times \left| \frac{s_B}{s_F} \right|^N \left| \frac{1 + \lambda s_F}{1 + \lambda s_B} \right|^L e^{\frac{N}{2\pi^2}(s_B^2 - s_F^2)}. \end{aligned} \quad (5.5)$$

It follows for the mean density of states (see equation (2.7))

$$\begin{aligned} \langle \rho(E) \rangle = & -\frac{1}{\pi} \operatorname{Im} \frac{dZ_1}{dE'} \Big|_{E'=E} \\ = & \frac{1}{\pi} \operatorname{Im} \frac{1}{2\pi i} \int_0^{i\infty} ds_B \oint ds_F \frac{e^{E(s_B - s_F)}}{s_B - s_F} \left| \frac{s_B}{s_F} \right|^N \left| \frac{1 + \lambda s_F}{1 + \lambda s_B} \right|^L e^{\frac{N}{2\pi^2}(s_B^2 - s_F^2)}. \end{aligned} \quad (5.6)$$

Differentiating with respect to E , we have after some straightforward transformations

$$\langle \rho'(E) \rangle = \frac{\pi^2}{2N} I_{NL}(\epsilon, \alpha) \bar{I}_{(N-1)L}(\epsilon, \alpha), \quad (5.7)$$

where

$$\epsilon = \frac{\pi}{\sqrt{2N}} E, \quad \alpha = \frac{\sqrt{N/2}}{\pi \lambda}, \quad (5.8)$$

and

$$I_{NL}(\epsilon, \alpha) = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} dx e^{2\epsilon x} \frac{(2x)^N}{(x + \alpha)^L} e^{x^2}, \quad (5.9)$$

$$\bar{I}_{NL}(\epsilon, \alpha) = \frac{1}{\pi i} \oint dy e^{-2\epsilon y} \frac{(y + \alpha)^L}{(2y)^{N+1}} e^{-y^2}. \quad (5.10)$$

From the definitions we immediately obtain the recursion relations

$$\begin{aligned} I'_{NL} &= I_{(N+1)L}, & (I_{NL}e^{2\epsilon\alpha})' &= I_{N(L-1)}e^{2\epsilon\alpha} \\ \bar{I}'_{NL} &= -\bar{I}_{(N-1)L}, & (\bar{I}_{NL}e^{-2\epsilon\alpha})' &= -\bar{I}_{N(L+1)}e^{-2\epsilon\alpha}, \end{aligned} \quad (5.11)$$

where the prime denotes differentiation with respect to ϵ .

For $L = 0$ we get in particular

$$I_{N0}(\epsilon, \alpha) = \frac{(-1)^N}{\sqrt{\pi}} e^{-\epsilon^2} H_N(\epsilon), \quad (5.12)$$

$$\bar{I}_{N0}(\epsilon, \alpha) = \frac{(-1)^N}{2^N N!} H_N(\epsilon), \quad (5.13)$$

where integral representations of the Hermite polynomials have been used (see e.g. Magnus *et al* 1966). Using the recursion relations we now calculate $\langle \rho(E) \rangle$ from equation (5.7) by repeated partial integration with the result

$$\langle \rho(E) \rangle = \frac{\pi}{\sqrt{2N}} \sum_{k=0}^{N-1} I_{kL}(\epsilon, \alpha) \bar{I}_{kL}(\epsilon, \alpha). \quad (5.14)$$

We have thus obtained a closed expression for the averaged density of states for arbitrary values of N and L . It is an easy matter to show, again using the recursion relations (5.11), that $\int_{-\infty}^{\infty} \langle \rho(E) \rangle dE = N$, as it should be.

For $L = 0$ equation (5.14) reduces to

$$\langle \rho(E) \rangle = \frac{\pi}{\sqrt{2N}} \sum_{k=0}^{N-1} [\psi_k(\epsilon)]^2, \quad (5.15)$$

where

$$\psi_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) e^{-x^2/2} \quad (5.16)$$

is an harmonic oscillator eigenfunction. This is identical with the well-known exact expression for the density of states of the Gaussian unitary ensemble, which in the limit of large N reduces to Wigner's semicircle law (Mehta 1991).

6. The strong coupling limit

Using the recursion relations (5.11) I_{NL} and \bar{I}_{NL} can be calculated from I_{N0} and \bar{I}_{N0} by repeated integration or differentiation, respectively. Since all integrations can be performed analytically, we have got an exact representation for the density of states for arbitrary L . Though this may be helpful for small values of L , it is not very useful for practical purposes, since one is usually interested in the limit $N, L \rightarrow \infty$ while the ratio $l = L/N$ remains finite.

In such a situation it suggests itself to solve the integrals (5.9) and (5.10) with help of saddle point techniques. This leads to a cubic saddle-point equation which

still can be solved exactly using Cardano's formula. The resulting equations are not very elucidating, however. Therefore we proceed in another direction and restrict the following discussion to the strong coupling limit $\lambda \gg 1$, or $\alpha \ll 1$. In the discussion we have to discriminate between the two situations $N > L$ and $N < L$.

(i) $N > L$

For this case we may replace $(x + \alpha)^L$ and $(y + \alpha)^L$ in the integrands by x^L and y^L , respectively, to obtain

$$I_{NL}(\epsilon, \alpha) = \frac{2^L}{\pi i} \int_{-i\infty}^{i\infty} dx e^{2\epsilon x} (2x)^{N-L} e^{x^2}, \quad (6.1)$$

$$\bar{I}_{NL}(\epsilon, \alpha) = \frac{2^{-L}}{\pi i} \oint dy e^{-2\epsilon y} (2y)^{-(N-L)} e^{-y^2}, \quad (6.2)$$

In the strong coupling limit the averaged density of states for a billiard system with N levels taken into account and L randomly distributed scatterers is thus the same as for a system with $N - L$ levels, and no scatterer at all. We are again in the random matrix regime.

Remember that already in the beginning we observed that the spectra of billiards with randomly distributed scatterers show random matrix behaviour, but at that point we considered the *weak* coupling limit $\lambda \ll 1$ (see the discussion following equation (2.25)).

We thus can note that for $N > L$ both in the weak and the strong coupling limit the averaged density of states shows random matrix behaviour.

(ii) $N < L$

Now we cannot replace any longer $(x + \alpha)^L$ and $(y + \alpha)^L$ in the integrands in equations (5.9) and (5.10) by x^L and y^L , since in this limit the integral for I_{NL} diverges, and that for \bar{I}_{NL} gives zero. In the limit $\alpha \ll 1$, on the other hand, the main contributions to the integrals come from regions $x \ll 1$ and $y \ll 1$, where the Gaussian cut-offs are not yet relevant. We may therefore replace e^{x^2} and e^{-y^2} by one, and solve the integrals by means of the residuum method with the result

$$\begin{aligned} I_{NL}(\epsilon, \alpha) &= 2^{N+1} \frac{\Theta(\epsilon)}{(L-1)!} \left(\frac{d}{dx} \right)^{L-1} \left(e^{2\epsilon x} x^N \right) \Big|_{x=-\alpha} \\ &= 2^{N+1} \alpha^{N+1-L} \Theta(z) (-1)^{N+1-L} e^{-z} L_{L-1}^{(N-L+1)}(z) \\ &= 2^{N+1} \alpha^{N+1-L} \Theta(z) \frac{N!}{(L-1)!} e^{-z} z^{L-N-1} L_N^{(L-N-1)}(z), \end{aligned} \quad (6.3)$$

$$\begin{aligned} \bar{I}_{NL}(z, \alpha) &= 2^{-N} \frac{1}{N!} \left(\frac{d}{dy} \right)^N \left(e^{-2\epsilon y} (y + \alpha)^L \right) \Big|_{y=0} \\ &= 2^{-N} \alpha^{L-N} L_N^{(L-N)}(z), \end{aligned} \quad (6.4)$$

where

$$z = 2\epsilon\alpha = E/\lambda. \quad (6.5)$$

$\Theta(z)$ is the Heaviside step function, and $L_n^{(\alpha)}(z)$ is a generalized Laguerre polynomial. (There are two conventions for the Laguerre polynomials found in literature, differing in the normalization. In this paper the definition of Magnus *et al* 1966 is adopted, where $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$.) It follows from equation (5.14) for the density of states

$$\langle \rho(E) \rangle = \frac{1}{\lambda(L-1)!} \Theta(z) e^{-z} \sum_{k=0}^{N-1} k! z^{L-k-1} L_k^{(L-k-1)}(z) L_k^{(L-k)}(z). \quad (6.6)$$

Equation (6.6) simplifies considerably in the limit $L \rightarrow \infty$, $N \rightarrow \infty$, with L/N remaining finite. Inserting expressions (6.3) and (6.4) for $I_{NL}(z, \alpha)$ and $\bar{I}_{NL}(z, \alpha)$, respectively, into equation (5.7) we obtain

$$\langle \rho'(E) \rangle = \frac{1}{\lambda^2} \frac{N!}{(L-1)!} \Theta(z) e^{-z} z^{L-N-1} L_N^{(L-N-1)}(z) L_{(N-1)}^{(L-N+1)}(z). \quad (6.7)$$

In terms of the function

$$y_n^{(\alpha)}(z) = e^{-\frac{z}{2}} z^{\frac{\alpha+1}{2}} L_n^{(\alpha)}(z), \quad (6.8)$$

equation (6.7) may be written as

$$\langle \rho'(E) \rangle = \frac{1}{\lambda^2} \frac{N!}{(L-1)!} \Theta(z) z^{-2} y_N^{(L-N-1)}(z) y_{N-1}^{(L-N+1)}(z). \quad (6.9)$$

The $y_n^{(\alpha)}$ obey the differential equation

$$y'' + \left(\frac{2n + \alpha + 1}{2z} - \frac{1}{4} + \frac{1 - \alpha^2}{4z^2} \right) y = 0. \quad (6.10)$$

Equation (6.10) is easily identified as the radial Schrödinger equation of the hydrogen atom, where $y_n^{(\alpha)}(z)/z$ is the radial part of the wave function. This suggests an approximation of $y_n^{(\alpha)}$ by means of the WKB method. In the present context it is sufficient to consider the solution in the classically allowed region. For this regime the WKB approximation yields (see e.g. section 9.3 of Morse and Feshbach 1953)

$$y_n^{(\alpha)}(z) = \frac{y_0}{\sqrt{q}} \cos \left(\int_{z_0}^z q \, dz - \frac{\pi}{4} \right), \quad (6.11)$$

where

$$q = \sqrt{\frac{2n + \alpha + 1}{2z} - \frac{1}{4} - \frac{\alpha^2}{4z^2}}, \quad (6.12)$$

and z_0, z_1 are the classical turning points given by

$$z_{0/1} = 2n + \alpha + 1 \pm \sqrt{(2n+1)(2n+2\alpha+1)}. \quad (6.13)$$

The replacement of $1 - \alpha^2$ by $-\alpha^2$ in going from equation (6.10) to equation (6.12) corrects for the singularity of the potential at $z = 0$, see the discussion in Morse und Feshbach 1953. (The same technique can be applied to derive the semi-circle law in a simple way from the exact expression (5.15), see chapter 3.2.3 of Stöckmann 1999; the procedure is more or less an elaboration of an idea developed in appendix A.9 of the book of Mehta (1991).)

Inserting approximation (6.11) into equation (6.9) we end up with

$$\langle \rho(E) \rangle = \frac{1}{\lambda\pi} \frac{1}{2z} \sqrt{4LN - (z - L - N)^2}, \quad z = E/\lambda, \quad (6.14)$$

where the classical turning points are given by

$$z_{0/1} = L + N \pm 2\sqrt{LN}. \quad (6.15)$$

Details of the derivation can be found in the appendix. The density of states thus changes dramatically if L surpasses N . For $L < N$ Wigner's semicircle law is found, and the eigenenergies are distributed between $-2\pi/N$ and $2\pi/N$. For $L > N$ on the other hand only positive eigenvalues are found, if λ is positive, in an energy window limited by λz_0 and λz_1 .

7. Discussion

We have obtained a surprisingly simple expression for the averaged density of states of a billiard with randomly distributed scatterers. The central ingredient was the idea to substitute the average over the scatterer positions in equation (2.16) by an weighted average, with the wave function amplitude probability density as the weight function. It was argued that both averages are equivalent. In view of the central importance of this procedure it seems appropriate to discuss the limitations of the approach.

(i) First, the impurities are considered as uncorrelated. In particular, it is not excluded that two impurities occupy the same site.

(ii) Second, wave functions belonging to different eigenvalues are considered as uncorrelated. This may pose a problem, since it is known from semi-classical quantum mechanics that there are correlations on energy scales of the order of \hbar/T , where T is the length of the shortest periodic orbit (see Gutzwiller 1990 for a review). On the other hand, these correlations vanish in the semi-classical limit on energy scales of the mean level spacing. It therefore seems legitimate to neglect correlations between different wave functions.

No problem, on the other hand, arises from the fact that there are spatial correlations for individual wave functions, as is well-known from the works of Berry (1977) and Fal'ko and Efetov (199). Since only the weight of the wave function amplitudes enters equation (2.18), spatial correlations are completely irrelevant.

The approximation performed in section 4 by substituting the billiard by a random matrix of finite rank is of another type. It has been applied to obtain a simple tractable model, but by this second step we have reduced our system to a mere caricature of a real

billiard system. In particular the information on the dimension of the billiard, which obviously is an important quantity for questions of localization and delocalization, is lost. (The information on the dimension is still present in equation (3.15), namely in the spectrum of the empty billiard which depends on the dimension via the mean density of states.)

This is why in the moment a comparison with literature results is not possible. In particular the work of Luttinger and Tao (1983) has to be mentioned in this respect, who calculated the density of states for the billiard with randomly distributed scatterers in the low energy limit. For a more detailed consideration of their results, we would have to go back to equations (3.15) to (3.17), perform the limit $N \rightarrow \infty$, and calculate the density of states for the true billiard system, and not a random matrix substitute only.

But the present results suggest that already in our toy model there is a localization-delocalization transition at $L = N$. For $L < N$ we are in the regime of delocalized wave functions obeying random-matrix behaviour. For $L > N$, on the other hand the wave functions become localized, giving rise to a completely changed density of states. In this respect the rank N of the matrix seems to take the role of the energy in the real billiard system.

For the moment, however, this conclusion must be considered as premature. Knowledge of the density of states is not sufficient to discriminate between localized and delocalized wave functions. For this we need additional information on the two-point correlation function, the inverse participation ratio and related quantities. The corresponding studies are under progress and will be published separately (Guhr and Stöckmann 2002).

If one compares the present approach with the non-linear σ model, a dramatic simplification is found. In the non-linear σ model one ends up with a supersymmetric integral over supersymmetric field variables which can be solved only within the zero-mode approximation. In our approach the very simple integral (4.9) is obtained instead, containing only one set of supersymmetric variables, which for the density of states even can be solved exactly. The same is true for *all* n -point correlation functions as will be shown in Guhr and Stöckmann (2002).

Even better, the assumption of a random distribution of point-like scatterers applied in this work is a much more realistic description of the situation found in mesoscopic systems than the assumption of a delta-correlated disorder potential assumed in the non-linear σ model.

It might be considered as a draw-back that the present derivation is based on two unproven conjectures, namely that (i) the wave function amplitudes in a chaotic billiard are Gaussian distributed, and that (ii) the eigenvalues in a chaotic billiard obey random matrix behaviour. On the other hand, there is such an overwhelming numerical evidence that both conjectures are true that one could equally well argue that both assumptions are even better founded than the assumption of Gaussian distributed matrix elements applied in random matrix theory.

From the point of view of an experimentalist it would be highly desirable, if all quantities of interest are available for systems with time-reversal symmetry as well. Though the calculations for this case are notoriously difficult, it should be worthwhile to undertake the effort. Experiments with microwaves on localization-delocalization transitions, pulse propagation etc. in disordered systems do already exist, as was mentioned in the introduction, and wait for their proper theoretical explanation.

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Appendix A. The density of states in the large L limit

To derive equation (6.14) for the density of states we start with equation (6.7),

$$\hat{\rho}'(z) = \frac{N!}{(L-1)!} e^{-z} z^{L-N-1} L_N^{(L-N-1)}(z) L_{N-1}^{(L-N+1)}(z). \quad (\text{A.1})$$

where we have introduced $z = E/\lambda$ as a new variable, and where $\hat{\rho}(z) dz = \rho(E) dE$. z is assumed to be positive in the following. Using elementary relations for the Laguerre polynomials, equation (A.1) may be transformed as follows:

$$\begin{aligned} \hat{\rho}'(z) &= -\frac{N!}{L!} e^{-z} \left[z^{L-N} L_N^{(L-N)} \right]' \left[L_N^{(L-N)} \right]' \\ &= -\frac{N!}{L!} e^{-z} \left[e^{\frac{z}{2}} z^{\frac{L-N-1}{2}} y_N^{(L-N)} \right]' \left[e^{\frac{z}{2}} z^{-\frac{L-N+1}{2}} y_N^{(L-N)} \right]', \end{aligned} \quad (\text{A.2})$$

where $y_N^{(L-N)}(z)$ is given by equation (6.8). It remains to determine the normalization constant y_0 . From the orthogonality relation for the Laguerre polynomials we have

$$\int_0^\infty \left[y_n^{(\alpha)}(z) \right]^2 \frac{dz}{z} = \int_0^\infty e^{-z} z^\alpha \left[L_n^{(\alpha)}(z) \right]^2 dz = \frac{(n+\alpha)!}{n!}. \quad (\text{A.3})$$

From the WKB approximation (6.11), on the other hand, we obtain

$$\int_0^\infty \left[y_n^{(\alpha)}(z) \right]^2 \frac{dz}{z} = \frac{y_0^2}{2} \int_{z_0}^{z_1} \frac{dz}{z q(z)} = \pi y_0^2, \quad (\text{A.4})$$

whence follows

$$y_0 = \left[\frac{1}{\pi} \frac{(n+\alpha)!}{n!} \right]^{\frac{1}{2}}. \quad (\text{A.5})$$

The sign has to be chosen positive to be in accordance with the usual definition of the Laguerre polynomials. Inserting now the WKB approximation for $y_n^{(\alpha)}(z)$ into equation (A.2) we have

$$\hat{\rho}'(z) = -\frac{1}{\pi} e^{-z} \left[e^{\frac{z}{2}} z^{\frac{L-N-1}{2}} \frac{1}{\sqrt{q}} \cos w \right]' \left[e^{\frac{z}{2}} z^{-\frac{L-N+1}{2}} \frac{1}{\sqrt{q}} \cos w \right]', \quad (\text{A.6})$$

where

$$q(z) = \sqrt{\frac{L+N+1}{2z} - \frac{1}{4} - \frac{(L-N)^2}{4z^2}}, \quad (\text{A.7})$$

and

$$w = \int_{z_0}^{z_1} q(z) dz - \frac{\pi}{4}. \quad (\text{A.8})$$

It follows

$$\begin{aligned} \hat{\rho}'(z) = & -\frac{1}{\pi z} \left[\left(\frac{1}{2} + \frac{L-N-1}{2z} - \frac{1}{2} \frac{q'}{q} \right) \frac{1}{\sqrt{q}} \cos w - \sqrt{q} \sin w \right] \\ & \times \left[\left(\frac{1}{2} - \frac{L-N+1}{2z} - \frac{1}{2} \frac{q'}{q} \right) \frac{1}{\sqrt{q}} \cos w - \sqrt{q} \sin w \right]. \end{aligned} \quad (\text{A.9})$$

The q' terms may be discarded since they are by an order of $1/L$ smaller as the other ones. For the same reason we may replace $L-N-1$ and $L-N+1$ by $L-N$. Averaging over the rapidly oscillating terms, we have

$$\hat{\rho}'(z) = -\frac{1}{2\pi z q} \left[\left(\frac{1}{2} + \frac{L-N}{2z} \right) \left(\frac{1}{2} - \frac{L-N}{2z} \right) + q^2 \right] \quad (\text{A.10})$$

Inserting expression (A.7) for q , we obtain

$$\begin{aligned} \hat{\rho}'(z) = & -\frac{1}{2\pi z q} \left[\frac{L+N}{2z} - \frac{(L-N)^2}{2z^2} \right] \\ = & \frac{1}{2\pi q} (q^2)' = \frac{1}{\pi} q', \end{aligned} \quad (\text{A.11})$$

where again terms of the order of $1/L$ have been neglected. It follows

$$\hat{\rho}(z) = \frac{1}{\pi} q, \quad (\text{A.12})$$

which is equivalent with equation (6.14). q. e. d.

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